

Dynamical Tunneling in Many-Dimensional Chaotic Systems

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We investigate dynamical tunneling in many dimensional systems using a quasi-periodically modulated kicked rotor, and find that the tunneling rate from the torus to the chaotic region is drastically enhanced when the chaotic states become delocalized as a result of the Anderson transition. This result strongly suggests that amphibious states, which were discovered for a one-dimensional kicked rotor with transporting islands [L. Hufnagel *et al.*, Phys. Rev. Lett. **89**, 154101 (2002)], quite commonly appear in many dimensional systems.

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According to the *semiclassical eigenfunction hypothesis*, each eigenfunction of a quantum system, whose classical counterpart exhibits mixed-type phase space, is expected to be localized on either regular or chaotic components of classical phase space in the semiclassical limit [1]. In a finite \hbar regime, however, the tunneling effect comes in and it hybridizes localized eigenfunctions together. This has invoked considerable interest in quantum tunneling in two or more degrees of freedom systems [2–6].

Since the tunneling effect is supposed to be exponentially small in general, one may regard tunneling merely as a correction to the semiclassical eigenfunction hypothesis. However, it was found that under a certain condition eigenfunctions are not necessarily localized on either regular or chaotic regions even when \hbar is much smaller than the area of the regular region. Thus it looks to be violating the semiclassical eigenfunction hypothesis [7, 8].

Such states are called *amphibious eigenstates* and their origin was discussed in Ref. [8]: They pointed out the importance of the relation between two time scales, the Heisenberg time T_H in the chaotic sea and the torus's life time T_L , to understand such exotic states [9]. Since the former is associated with the localization length in the chaotic sea, the argument on the two competing time scales can be rephrased using the interplay between dynamical localization [10] and dynamical tunneling.

A more direct explanation for the “flooding” of the wavefunction [7, 8] can be given based on the time-domain semiclassical picture [11]. In particular, the theory of complex dynamical systems tells us that there are exponentially many complex orbits which connect the torus and chaotic regions. It could further be proved that they (i) act as tunneling orbits when they stay in the torus region, and (ii) behave as if they are real orbits after reaching the chaotic sea [11]. In other words, dynamical tunneling and dynamical localization are controlled by the same orbits with amphibious characters, i.e., both regular and chaotic ones. Therefore, destructive interference among the semiclassical orbits causing the suppression of classical diffusion [12] simultaneously

inhibits chaotic tunneling from the torus to the chaotic sea. On the contrary, the attenuation of dynamical localization and the recovery of the chaotic tunneling occur simultaneously [13].

Amphibious states have been considered to appear in quite a specific situation where the system has accelerator modes [7, 8]. However, the accelerator modes are introduced just to prepare extremely large localization lengths [8]. The system with large localization lengths can also be realized in many degrees of freedom systems according to the correspondence of the dynamical localization to the Anderson localization [14–17]. This naturally leads to a conjecture: *The appearance of amphibious states is a common feature in many-degrees of freedom systems.* This, however, needs to be scrutinized, since Bäcker *et al.*'s argument in Ref. [8] provides only a sufficient condition of the absence of amphibious states under the small localization lengths, and no condition that ensures the amphibious states is known. In this Letter, we will provide numerical evidence of the conjecture.

We examine a one degree of freedom kicked rotor whose phase space (q, p) is divided into a torus region $p \lesssim b$ and a chaotic sea $p \gtrsim b$, which are connected by dynamical tunneling. To introduce many-dimensionality in effect, the system is described by the kicked Hamiltonian with quasi-periodic modulations [15, 19]:

$$H = T(p) + V(q) \sum_n \delta(t - n) + \epsilon g(p) \frac{1}{M} \sum_{j=1}^M \cos(\omega_j t) \sum_n \delta(t - n - 0), \quad (1)$$

where $V(q) = k \cos(2\pi q)/(4\pi^2)$, and modulation frequencies $\{\omega_j/(2\pi)\}_{j=1}^M$ are irrational and non-resonant with each other [20]. $T(p)$ and $g(p)$ are specified in the following. We impose periodic boundary conditions $0 \leq q < 1$ and $-W_p/2 \leq p < W_p/2$ [21].

Before studying the mixed case, we first examine the fully chaotic case to identify insulator and metallic phases in the parameter space of ϵ , with a sufficiently large value of W_p . We choose $T(p) = s(p - b)^2/2$ and $g(p) = p - b$.

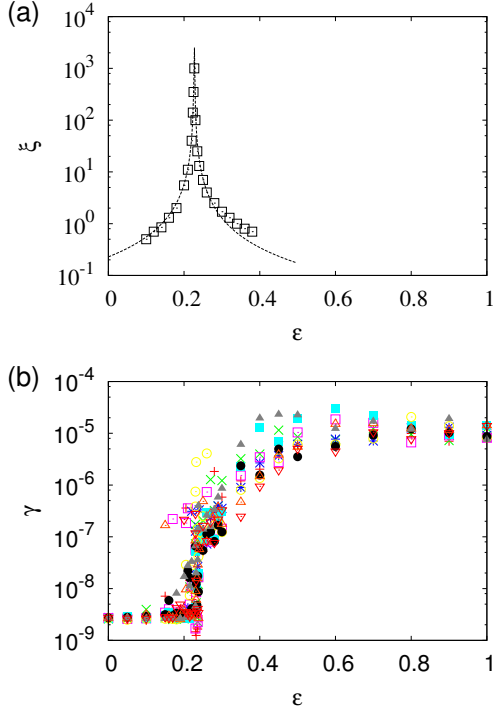


FIG. 1: (Color online) (a) The scaling parameter ξ , which is proportional to the localization length and the inverse of the diffusion constant in the insulator and metallic phases, respectively, as a function of the strength of the modulation ϵ in our model [Eq. (1)]. $T(p)$ and $g(p)$ are chosen so as to make the whole phase space chaotic (see the main text). ξ diverges at the critical point $\epsilon_c \simeq 0.227$. A thin curve indicates $\xi = a|\epsilon - \epsilon_c|^{-\nu}$ with $\nu = 1.55$ and $a = 2.4 \times 10^{-2}$. The parameters are set to be $k = 2$, $s = 4 + \sqrt{3}/10$, $b = 0$, $M = 2$, $\omega_1 = 2\pi\sqrt{5}$, $\omega_2 = 2\pi\sqrt{13}$, $W_p = 1000$, and Planck's constant $\hbar = 0.2$. (b) The decay rate γ of P_n^T [Eq. (3)] as a function of ϵ . Different marks correspond to ten different values of s for a small interval $4.17 < s < 4.19$. For $\epsilon \ll \epsilon_c$, chaotic tunneling is almost forbidden, i.e., $\gamma \sim 0$ [18]. Around $\epsilon \sim \epsilon_c$, γ exhibits strong fluctuation, which becomes smaller for larger ϵ . The parameters for Eq. (2) are $b = 5.6 - W_p/2$, $\tilde{b} = b + 0.2$, $\omega = \sqrt{3}$ and $\beta = 50$. The initial state $|\psi_0\rangle$ satisfies the EBK condition whose center of the momentum is at $p = b - 0.6$ (see, Appendix of Ref. [13]).

In the absence of modulation, i.e., $\epsilon = 0$, the mapping derived by Eq. (1) can be reduced to the standard mapping whose nonlinearity parameter is $K = sk$. In the following, we examine the case that the nonlinearity is sufficiently large $K \sim 8$, where the phase-space is mostly filled with a chaotic sea [22]. When ϵ is non-zero, this model is essentially the same as Casati *et al.*'s quasi-periodic kicked rotor, which is equivalent with a $M + 1$ dimensional tight-binding model with pseudo-disorder and is shown to exhibit the Anderson transition with $M = 2$ [15, 17]. We also examine the case $M = 2$ and numerically confirmed that our model is in insulating and metallic phases at $\epsilon < \epsilon_c$ and $\epsilon > \epsilon_c$, respectively,

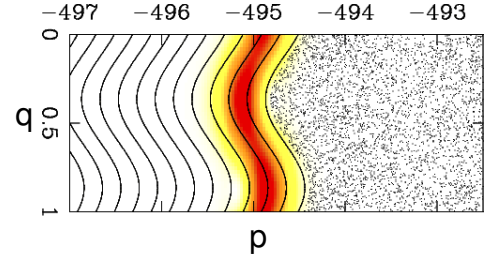


FIG. 2: (Color online) Poincaré section around the torus region of the tunneling degrees of freedom under the absence of modulation (i.e., $\epsilon = 0$). The torus region $p \lesssim b$ and the chaotic sea $p \gtrsim b$ are seen to be sharply divided. The density plot of the Husimi function of the initial torus state, which is constructed by EBK quantization, is superposed. The parameters are same as in Fig. 1(b), except $s = 4 + \sqrt{3}/10$.

where $\epsilon_c \simeq 0.227$ is the metal-insulator transition point. Using the finite size scaling technique [17, 23, 24], we obtained numerical evidence of the Anderson transition in the chaotic sea. The details will be reported elsewhere [25]. We show how the scaling parameter ξ [24], which is proportional to the localization length and the inverse of diffusion constant in the insulating and metallic phase, respectively, depends on ϵ in Fig. 1(a).

Next, we introduce a torus region in $p \lesssim b$ with keeping untouched the chaotic nature of the region $p \gtrsim b$ (see, Fig. 2), using the following kinetic term [13]

$$T(p) \equiv \frac{s}{2}(p - b)^2 \theta_\beta(p - b) + \omega(p - b), \quad (2)$$

where $\theta_\beta(x) = [1 + \tanh(\beta x)]/2$ is a smoothed step function with a smoothing parameter β [26]. The region $p \lesssim b$ is filled with tori because $T(p)$ is effectively linear there. At the same time, we employ the modulation term $g(p) \equiv (p - \tilde{b})\theta_\beta(p - \tilde{b})$, where \tilde{b} is slightly larger than b . This makes the torus region almost independent of the modulation. Hence, only dynamical tunneling induces transitions between the torus region and the chaotic sea.

Tunneling leakage from the torus to chaotic regions is monitored by the integration of $|\langle p|\psi_n\rangle|^2$ for the whole torus region [27]

$$P_n^T \equiv \int_{-W_p/2}^b |\langle p|\psi_n\rangle|^2 dp, \quad (3)$$

where $|\psi_n\rangle$ denotes the state vector at time step n . We prepare the system initially to be in a torus state $|\psi_0\rangle$, which satisfies the Einstein-Brillouin-Keller (EBK) quantization in the torus region (see, Fig. 2). Time evolution of P_n^T strongly depends on ϵ , as shown in Fig. 3. When ϵ is far below ϵ_c , P_n^T keeps almost unity within, say, 10^6 steps. As shown in Fig. 4(a), the corresponding wave function in the momentum representation exhibits dynamical localization in the chaotic sea. As was shown in [28], and will be reported in [25], there appear exponentially many complex orbits connecting the torus and

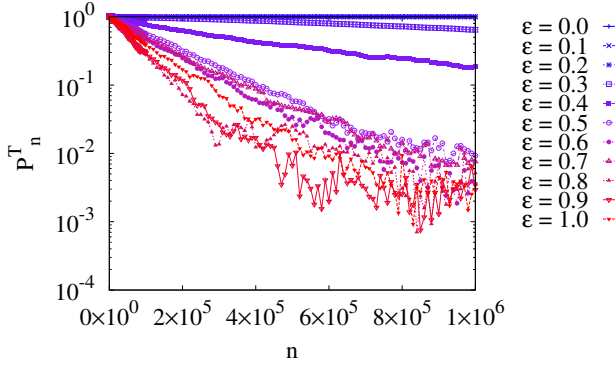


FIG. 3: (Color online) The survival probability P_n^T of a torus state as a function of the number of kicks n . The parameters are the same as in Fig. 2. When the chaotic sea is in the insulating phase $\epsilon < \epsilon_c$, the tunneling decay is almost forbidden. For $\epsilon \geq 0.3$ ($> \epsilon_c$), P_n^T exhibits exponential decay. Around the transition point, the decay of P_n^T strongly depends on ϵ , and the decay is too slow to determine whether or not P_n^T obeys the exponential law.

chaotic region, nevertheless such “flooding” of complex orbits are suppressed because of the destructive interference in the chaotic region.

For larger ϵ , chaotic tunneling recovers (see, Fig. 3). Figure 1(b) shows the ϵ -dependence of decay rate γ , which is obtained by fitting of P_n^T . Around ϵ_c , γ changes significantly with large fluctuations. A strong correlation between ϵ dependences of γ and ξ is evident. Indeed, when γ is significantly different from zero, the classical diffusion in the chaotic sea $p \gtrsim b$ is recovered, as shown in Fig. 4(b). This strongly suggests that chaotic tunneling recovers when the chaotic sea is in the metallic phase. Note that the localized component on the torus region almost disappears at $n = 10^6$, and the tails of wavefunction is clearly given as Gaussian (see inset of Fig. 4(b)).

We remark on the fluctuation of γ (see, Fig. 1(b)). Around ϵ_c , γ strongly depends on s as well as ϵ . A possible origin of the fluctuation is the resonance induced by near degeneracies among approximate quasienergies of torus and chaotic states [13]. Such a strong fluctuation against parameter variations is a common feature of dynamical tunneling in nonintegrable systems [2]. For smaller ϵ ($\ll \epsilon_c$), the resonances are ineffective [29]. In contrast, the effect of the resonances becomes prominent around ϵ_c . This is because the effective density of states around the torus state become larger due to the exponential growth of the localization length in the chaotic sea (see, Fig. 1(a)). For much larger ϵ ($\gg \epsilon_c$), the fluctuation becomes smaller. This is supposed to be due to the completion of the transition to the metallic phase. In other words, the effective density of states around the torus state become so large, the fluctuations induced by the resonances are averaged out.

So far, we have focused on the case that W_p is suffi-

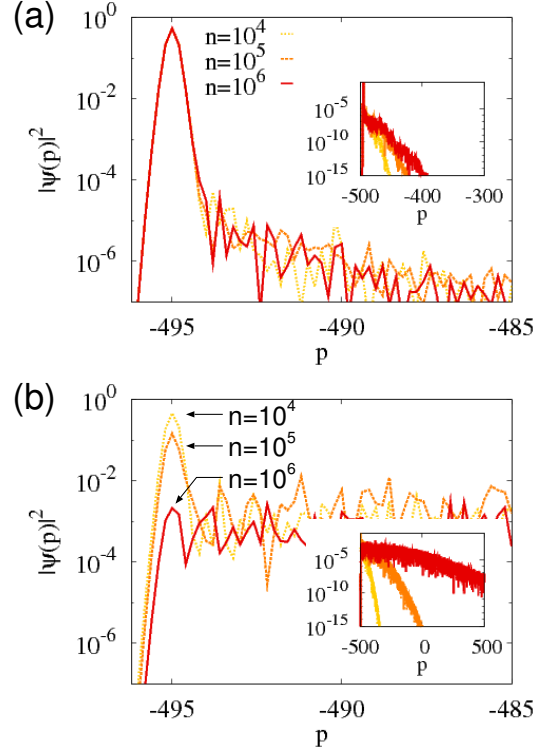


FIG. 4: (Color online) Snapshots of the momentum distribution $|\langle p | \psi_n \rangle|^2$. (a) In the insulating phase ($\epsilon = 0.2$), $|\langle p | \psi_n \rangle|^2$ remains almost unchanged in the torus region and dynamical localization occurs in the chaotic sea. (b) In the metallic phase ($\epsilon = 0.8$), $|\langle p | \psi_n \rangle|^2$ decays in the torus region and the envelope of $|\langle p | \psi_n \rangle|^2$ in the chaotic sea around the torus region is almost unchanged. However, as is seen in the inset of (b), the diffusion toward the uniform distribution in the momentum space is not completed for $n < 10^6$.

ciently large. As for the “phase transition” between the suppression and restoration of chaotic tunneling, our numerical result suggests that the convergence for the “thermodynamic” limit $W_p \rightarrow \infty$ is quite fast. In this idealized limit, one may regard that the delocalized chaotic sea plays a role of “particle bath”.

To be precise, however, the transition from the suppression of chaotic tunneling to the recovery is not sharp, but rather smooth, as seen in Fig. 1(b). This is because the recovery occurs even in the insulating phase with sufficiently large localization length due to the Bäcker *et al.*’s condition $T_L \simeq T_H$ [8], which determines the border between the suppression and the recovery.

Furthermore, the crossover between the suppression and the restoration of chaotic tunneling occurs even when W_p is rather small. If ϵ is sufficiently larger than ϵ_c , the time evolution of P_n^T obeys the irreversible decay for a short time period, and after that, exhibits erratic oscillations (see, Fig. 5). Although the physical picture based on the thermodynamic limit is inapplicable anymore, our numerical result indicates the presence of a considerable

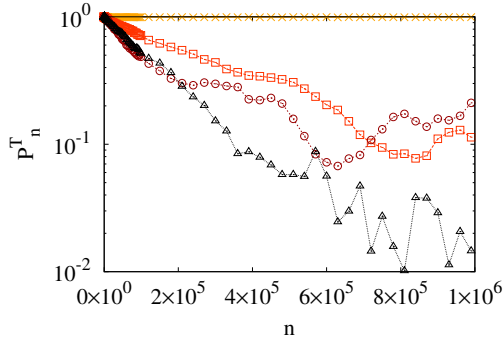


FIG. 5: (Color online) Evolution of P_n^T for the case of smaller $W_p = 30$. We choose $\epsilon = 0.5$ ($> \epsilon_c$). To ensure the spectrum of the system in the delocalized “phase” to be discrete one, the modulation frequencies are chosen to be rational [15], i.e., $\omega_1/(2\pi) = 682/305$ and $\omega_2/(2\pi) = 11/3$ (\times), $119/33$ (\square), $649/180$ (\circ), and $4287/1189$ (\triangle). Other parameters are the same as in Fig 3.

number of amphibious states, i.e., eigenstates that have significant overlap with both torus and chaotic regions.

We have numerically investigated the stability of torus states surrounded by chaotic seas in many degrees of freedom systems. If the coupling strength exceeds a critical value at which the Anderson transition occurs in the chaotic region, the nature of dynamical tunneling drastically changes. The result of the smaller W_p case may give rise to a reexamination of quantization condition of many dimensional mixed systems. Although the quantization of tori and chaotic seas can be carried out separately in the semiclassical limit, this seems not to be the case even when the size of Planck’s constant is considerably smaller than the area of a torus region, due to the emergence of amphibious states. The “nonseparability” among regular and chaotic region certainly has been a problem of tunneling in nonintegrable systems. Our result suggests that the occurrence of the nonseparability, i.e., the emergence of amphibious states, is the rule rather than the exception in many-dimensional mixed systems.

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